Journal of Statistical Physics, Vol. 37, Nos. 1/2, 1984

Description of a Class of Markov Processes "Equivalent" to K-Shifts

S. Martínez[1] **and E. Tirapegui**[2]

Received April 12, 1983; revised January 17, 1984

It results from recent works of Prigogine and collaborators that one can construct a nonunitary operator which realizes an "equivalence" between the positive actions of a reversible dynamical system and an irreversible Markov process going to equilibrium. We consider here this construction and we prove that (a) for K-shifts the transition probability of the associated Markov process is concentrated in the stable manifold of the transformed point by the shift with a point mass concentrated on the deterministic trajectory; and (b) for Bernoulli shifts the measures which go to equilibrium are the same for the deterministic system and the Markov process.

KEY WORDS: K-systems; Bernoulli shifts; Markov processes; irreversibility; stable manifolds.

1. INTRODUCTION

A central problem in classical statistical mechanics is to find methods to associate (if possible without reduction of description) stochastic processes going to equilibrium to the reversible evoluations of classical mechanics. Recently Courbage, Misra, and Prigogine[(1-4)] have proposed a method that allows one to construct a Markov process "equivalent" to the reversible evolution of some dynamical systems, and this without loss of information. It is this construction that will interest us here. The problem is as follows: given a reversible dynamical system (Ω, α, μ, T) one looks for an irreversible

[1] Departamento de Matemáticas y Ciencias de la Computación, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago, Chile. Supported in part by Servicio de Desarrollo de la Investigación (Universidad de Chile) and by Coopération Culturelle et Technique of France in Chile.

[2] Instituut voor Theoretische Fysica, Universiteit Leuven, B-3030 Leuven, Belgium. On leave of absence from Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile.

0022-4715/84/1000-0173$03.50/0

Markov system, i.e., a μ-preserving Markov process (Ω, α, μ, Q), which goes to equilibrium and is "equivalent" to the T evolution. More precisely one asks for the operator $U^*f = f \circ T^{-1}$ and the evolution operator Q^* of the Markov system to be related by a positive nonunitary operator $\Lambda : L^2(\mu) \to L^2(\mu)$, such that Λ^{-1} is defined in a dense subspace and $\Lambda U^* = Q^* \Lambda$.

In Section 2 we give the general mathematical frame (following Refs. 5 and 6). We introduce the notion of $\mathscr{D}$-deformation and prove its essential properties. Then we recall the result that for a K-automorphism the operator Λ can be taken[4] as $\Lambda = \sum_{n \in Z} \lambda_n E_n + P_0$, where $(\lambda_n)_{n \in Z} \subset (0, 1)$ is a strictly decreasing sequence, P_0 the projector in the space of constant functions, and $E_n = E^{\alpha_n} = E^{\alpha_n} - E^{\alpha_{n-1}}$, with $\alpha^n = T^n \alpha_0$ and α_0 is the σ-algebra defining the K-automorphism.

In Section 3 we describe the previous Markov process in the case of K-shifts generalizing our previous results.[7,8] We compute explicitly the transition probability Q_W; we prove that if $\nu_\infty = \lim_{n \to \infty} \lambda_n^{-1} \lambda_{n+1} > 0$ it has a point mass concentrated on the deterministic trajectory and moreover that it is concentrated in the stable manifold (this concept being a generalization from hyperbolic differential dynamical systems) of the transformed point of the shift, i.e., $Q_W(x, X_\sigma^{\text{st}}(\sigma x)) = 1$, where $X_\sigma^{\text{st}}(y) = \{z \in S^Z : d(\sigma^n x, \sigma^n y) \to_{n \to \infty} 0\}$, with σ the shift transformation and S the alphabet (Theorem 2). In the case of Bernoulli shifts we decompose $Q_W(x, \cdot)$ in the fibers of the stable manifold of σx and we show (Proposition 1) that in each fiber it is distributed uniformly with respect to the restriction of the Bernoulli measure.

Finally in Section 4 we study the measures going to equilibrium (weak convergence). In Theorem 3 we prove that in the case of Bernoulli shifts the set of probability measures converging to the invariant measure is the same for the dynamical system and for the associated Markov process. But the operator Q_Λ induced by Λ acting on this set is not bijective.

2. DYNAMICAL SYSTEMS, MARKOV PROCESSES, DEFORMATIONS

2.1. Dynamical Systems and Markov Processes

Let (Ω, α) be a measurable space (α a σ-algebra over Ω) and $T: \Omega \to \Omega$ a measurable transformation. Then T acts on the space $\rho(\Omega, \alpha)$ of probability measures: $T\tau(A) = \tau(T^{-1}A)$, $\forall A \in \alpha$. If $\mu = T\mu$ we say μ is T-invariant and (Ω, α, μ, T) is a dynamical system. In the rest of this article μ is a fixed T-invariant measure. In the space M of real measurable functions we define the operator $U: M \to M$ by $Uf = f \circ T, f \in M$. If $f \geqslant 0$ then $Uf \geqslant 0$ (U is positive on M) and $U1 = 1$ (U preserves the constants). Let $\mathscr{L}^p(\mu) = \{f \in M : \|f\|_p =$

$(\int |f|^p \, d\mu)^{1/p} < \infty\}$, $p \geqslant 1$, and let $(L^p(\mu), \| \|_p)$ be the Banach space whose elements are the equivalence classes (for the relation μ-a.e.) of $\mathscr{L}^p(\mu)$. In $L^2(\mu)$ the norm $\| \|_2$ (that we denote $\| \|$) is induced by the scalar product $\langle f, g \rangle = \int fg \, d\mu$. The operator U acts on $L^p(\mu)$ and the T-invariance of μ implies $\| Uf \|_p = \| f \|_p$, $\forall p \geqslant 1$. In $L^2(\mu)$ the adjoint operator U^* is positive and preserves the constants. We say (Ω, α, μ, T) is mixing if $\langle U^n f, g \rangle \to_{n \to \infty} \langle f, 1 \rangle \langle 1, g \rangle$, $f, g \in L^2(\mu)$. Mixing implies ergodicity: $Uf = f$, $f \in L^2(\mu) \Rightarrow f = \text{const} = \langle f, 1 \rangle$. If T^{-1} exists and is measurable (i.e., T is bijective and bimeasurable), we shall say (Ω, α, μ, T) is reversible. Then U is invertible, $U^{-1} = f \circ T^{-1}$, and in $L^2(\mu)$ is unitary. An equivalence relation can be defined between dynamical systems[6] which preserves ergodicity and mixing. We shall only consider Lebesgue space, i.e., (Ω, α, μ) is isomorphic to the space $([0, 1], \alpha_{\mathscr{L}}, \mu_{\mathscr{L}})$, where $\alpha_{\mathscr{L}}$ and $\mu_{\mathscr{L}}$ are, respectively, the Lebesgue σ-algebra and a Lebesgue–Stieltjes measure. We shall say a reversible dynamical system is a K-system if there exists a sub-σ-algebra α_0 of α such that

$$\alpha_0 \subset T\alpha_0 \tag{2.1$'$}$$

$$\alpha_{-\infty} = \bigcap_{n \in Z} T^n \alpha_0 =_{(\mu)} \{\phi, \Omega\} \tag{2.1$''$}$$

i.e., $\alpha_{-\infty}$ is trivial,

$$\alpha_{\infty} = \bigvee_{n \in Z} T^n \alpha_0 \underset{(\mu)}{=} \alpha \tag{2.1$'''$}$$

where $=_{(\mu)}$ stands for equality except for sets of measure zero. K-systems are mixing and invariant under equivalence. Putting $\alpha_n = T^n \alpha_0$, $n \in Z$, one has

$$\{\phi, \Omega\} \underset{(\mu)}{=} \alpha_{-\infty} \subset \cdots \subset \alpha_n \subset \cdots \alpha_{\infty} = \alpha \tag{2.2}$$

$$T\alpha_n = \alpha_{n+1}, \qquad n \in Z \tag{2.3}$$

Let $L^2(\mu \mid \alpha_n) = \{f \in L^2(\mu) : f \text{ is } \alpha_n \text{ measurable}\}$ and let K_0 be the space of constant functions, then

$$K_0 = L^2(\mu \mid \alpha_{-\infty}) \subsetdot \cdots \subsetdot L^2(\mu \mid \alpha_n) \subsetdot \cdots \subsetdot L^2(\mu \mid \alpha_{\infty}) = L^2(\mu) \tag{2.2$'$}$$

where $G_1 \subsetdot G_2$ if G_1 is a closed linear subspace of G_2, and $U^{-1} L^2(\mu \mid \alpha_n) = L^2(\mu \mid \alpha_{n+1})$. Let $H_n = L^2(\mu \mid \alpha_n) \, \theta L^2(\mu \mid \alpha_{n-1})$. One has $H_n \neq \{0\}$ and $U^{-1} H_n = H_{n+1}$, $K_0 U^{-1} = U^{-1} K_0 = K_0$, $I = \sum_{n \in Z} E_n + P_0$, where E_n (respectively, P_0) is the orthogonal projection on H_n (respectively K_0). The orthogonal projection on $L^2(\mu \mid \alpha_n)$ is the conditional expectation value E^{α_n} over α_n, and we note $R_n = E^{\alpha_n}$ this operator which is positive. Then $E_n =$

$R_n - R_{n-1}$ and one has [in $L^2(\mu)$] $E_{n+1}U^{-1} = U^{-1}E_n$, $U^{-1}P_0 = P_0U^{-1} = P_0$. Due to (2.2) $R_nf \to_{n\to\infty} f$, $R_nf \to_{n\to-\infty} P_0f$, μ-a.e. in $\mathscr{L}^2(\mu)$ and for any $f \in \mathscr{L}^2(\mu)$.

We consider now shift dynamical systems. The finite set $S = \{1,..., s\}$ will be the alphabet of the shift and the space the product space $X = S^Z$ with the product σ-algebra $\mathscr{B} = P(S)^Z$ where $P(S)$ is the class of subsets of S. $\mathscr{B}$ is the Borel σ-algebra of the compact metric space (X, d) where $d(x, y) = \sum_{n\in Z} 2^{-|n|} d_s(X_n, y_n)$, with $x = (x_n)_{n\in Z} \in X$, $y = (y_n)_{n\in Z} \in X$, $d_s(s', s'') = 0$ if $s' = s''$, $d_s(s', s'') = 1$, if $s' \neq s''$. $\mathscr{B}$ is generated by the countable semialgebra of cylinders ζ: $B(k, m)$ $(i_0,..., i_l) \in \zeta$ if $B(k, m)(i_0,..., i_l) = \{x \in X: x_k = i_0,..., x_m = i_l\}$, for $k, m \in Z$, $l = m - k$, $i_0,..., i_l \in S$ fixed. A basis of neighborhoods $\zeta(x)$ of $x \in X$ is the set of cylinder neighborhoods centered on $x = B(k,m)(x) = \{y \in X: y_k = x_k,..., y_m = x_m\}$.

The shift transformation $\sigma: X \to X$, $x \to \sigma x$, where $(\sigma x)_n = x_{n+1}$, $n \in Z$, is a homeomorphism, and consequently bimeasurable. Any σ-invariant measure μ defines a reversible dynamical system $(\Omega, \alpha, \mu, \sigma)$ called a shift dynamical system. A class of σ-invariant measures are the Bernoulli measures: let $\pi = (\pi_1,..., \pi_s)$ be a probability vector ($\pi_i > 0$, $\sum_{i\in S} \pi_i = 1$) and define $\mu_\pi(B(k, m)) = \pi_{i_0} \cdots \pi_{i_l}$; then $(X, \mathscr{B}, \mu_\pi, \sigma)$ is a Bernoulli shift. In the shift $(X, \mathscr{B}, \mu, \sigma)$ the cylinders $X_i = \{x \in X: x_0 = i\}$, $i \in S$, define a partition α_0 over X: $\alpha_0 = \{X_i | i \in S\}$. We note $\mathscr{B}_n = \bigvee_{-\infty}^{n} \sigma^i\alpha_0$ the σ-algebra generated by $\{\sigma^i\alpha_0,\ i \leqslant n\}$, $\mathscr{B}_\infty = \mathscr{B}$, i.e., α_0 is a generating partition. If $\mathscr{B}_{-\infty} = \bigcap_{n\leqslant 0} \mathscr{B}_n$ is trivial the K-system $(X, \mathscr{B}, \mu, \sigma)$ is called a K-shift. For shifts a version of the projectors $R_n = E^{\mathscr{B}_n}$ can be chosen verifying pointwise for any cylinder $B = B(k,m)$

$$R_n\xi_B = \xi_B, \qquad n \geqslant -k \tag{2.4}$$

$$R_n\xi_B = \xi_{B(-n,m)} R_n \xi_{B(k,-(n+1))}, \qquad -m \leqslant n < -k \tag{2.4'}$$

$$R_n\xi_B \xrightarrow[n\to-\infty]{} \mu(B), \qquad |R_n\xi_B| \leqslant 1 \tag{2.4''}$$

where ξ_A is the characteristic function of the set A. In the case of Bernoulli shifts we can furthermore specify the version of R_n as

$$R_n\xi_{B(k,m)(x)} = \mu_\pi(B(k,m)(x)), \qquad n < -m \tag{2.4'''}$$

We consider now Markov processes. A transition probability from the space (Ω, α) to the space (Ω', α') is a function $Q: \Omega \times \alpha' \to [0, 1]$, $(\omega, A') \to Q(\omega, A')$, satisfying (a) $Q(\omega, \cdot)$ is a probability measure on (Ω', α'), $\omega \in \Omega$; and (b) $Q(\cdot, A')$ is α-measurable $\forall A' \in \alpha'$. If $(\Omega, \alpha) = (\Omega', \alpha')$ the transition probability defines a stationary Markov process (Ω, α, Q).[5] If $T: \Omega \to \Omega$

is a measurable transformation it induces a transition probability $Q_T(\omega, \{T\omega\}) = 1$. Q acts over measures Q: $\rho(\Omega, \alpha) \to \rho(\Omega', \alpha')$ as

$$(Q\tau)(A') = \int \tau(d\omega)\, Q(\omega, A'), \qquad A' \in \alpha', \tau \in \rho(\Omega, \alpha) \tag{2.5}$$

and over the space of bounded functions $Q: \mathscr{M}_b(\Omega', \alpha') \to \mathscr{M}_b(\Omega, \alpha)$ as

$$(Qf)(\omega) = \int f(\omega')\, Q(\omega, d\omega'), \qquad \omega \in \Omega, \quad f \in \mathscr{M}_b(\Omega', \alpha') \tag{2.6}$$

If we note $\xi_{A'}$ the characteristic function of $A' \in \alpha'$, then $Q(\omega, A') = (Q\xi_{A'})(\omega)$. For $\tau \in \rho(\Omega, \alpha)$, $f \in \mathscr{M}_b(\Omega, \alpha)$ we define $\tau(f) = \int \tau(d\omega) f$; then $(Q\tau) f_1 = \tau(Qf_1)$, $f_1 \in \mathscr{M}_b(\Omega', \alpha')$. Let $\mu \in \rho(\Omega, \alpha)$, (Ω, α, μ) Lebesgue; then one can extend Q to $\mathscr{L}^1(\mu)$. If $Q\mu = \mu'$, $\int Qf\, d\mu = \int f\, d\mu'$, $f \in \mathscr{L}^1(\mu')$. The operator $Q: \mathscr{L}^1(\mu') \to \mathscr{L}^1(\mu)$ is positive and preserves the integral, $Q1 = 1$.

Since $(Qf)^2 \leqslant Q(f^2)$ one has $\int (Qf)^2\, d\mu \leqslant \|f\|^2$; consequently Q: $L^2(\mu') \to L^2(\mu)$ is positive, preserves the constants, and $\|Q\| = 1$. The same holds for the adjoint Q^*. Reciprocally, from an extension of Propositions V-4-2 and V-4-4 of Ref. 5, and by using the Lebesgue structure of the probability space, one has the following lemma.

Lemma 1. Let (Ω, α, μ) and (Ω', α', μ') be Lebesgue, and $V: L^2(\mu) \to L^2(\mu')$ a positive bounded linear operator such that $V1 = 1$, $V^*1 = 1$. Then there exists a transition probability Q_{V^*} from (Ω, α) to (Ω', α') with $Q_{V^*}\mu = \mu'$ and such that for any $f \in L^2(\mu')$ the function $Q_{V^*} f$ belongs to the equivalence class $V^* f$ (then $Q_{V^*} = V^*: L^2(\mu') \to L^2(\mu)$ and $Q_{V^*}{}^* = V$).

We shall say that V^* induces the transition probability (which is not unique) Q_{V^*}. In an analogous way one can construct a transition probability Q_V induced by V from (Ω', α') to (Ω, α), $Q_V \mu' = \mu$. If $(\Omega, \alpha, \mu) = (\Omega', \alpha, \mu')$ these transition probabilities induce Markov processes preserving the measure μ. If Q_{V^*} and Q'_{V^*} are two transition probabilities induced by V^* one has $Q_{V^*}\xi_{A'}(\omega) = Q'_{V^*}\xi_{A'}(\omega)$ μ-a.e. and since the space are Lebesgue there exists a set $\Omega_1 \in \alpha$, $\mu(\Omega_1) = 1$, such that $Q_{V^* A'} = Q'_{V^* A'}$ on Ω_1, $\forall A' \in \alpha'$.

Let Q be a transition probability in (Ω, α), i.e., from (Ω, α) to itself, which preserved μ: $Q\mu = \mu$, and (Ω, α, μ) Lebesgue. We shall say (Ω, α, μ, Q) is a Markov system. An initial state at time zero will be a measure $\tau_0 \in \rho(\Omega, \alpha)$ evolving by (2.5), i.e., at time $n > 0$ the state will be the measure $\tau_n = Q^n \tau_0$. If τ_0 has a probability density $f_0 \in L^2(\mu)$ with respect to μ, then at time $n > 0$ this probability density will be $f_n = Q^{*n} f_0$, where $Q^*: L^2(\mu) \to L^2(\mu)$ is the adjoint in $L^2(\mu)$ of the operator Q defined by (2.6). Obviously the dynamical system (Ω, α, μ, T) defines a Markov system (Ω, α, μ, Q) where the transition Q is induced by U, $Uf = f \circ T$, and

$Q(\omega, \{T\omega\}) = 1$. One has $Q = U$ on $L^2(\omega)$ and in the reversible case $Q_{U^*} = U^* = U^{-1}$ is the evolution operator (we shall note $Q = Q_T$).

Definition 1. We shall say the Markov system (Ω, α, μ, Q) goes to equilibrium if $a_n(f_0) = \|Q^{*n}f_0 - 1\| \downarrow_{n\to\infty} 0$ for any density $f_0 \in L^2(\mu) - K_0$. The notation $a_n \downarrow_{n\to\infty} a$ means that the sequence of real numbers $(a_n)_{n\geqslant 1}$ converges to a and is strictly decreasing.

The property of going to equilibrium implies

$$\|Q^{*n}f - \langle f, 1\rangle\| \xrightarrow[n\to\infty]{} 0, \qquad f \in L^2(\mu)$$

and also that Q^* is mixing:

$$\langle Q^{*n}f, g\rangle = \langle f, Q^n g\rangle \xrightarrow[n\to\infty]{} \langle f, 1\rangle\langle 1, g\rangle, \qquad f, g \in L^2(\mu)$$

which implies the ergodicity of Q^*, i.e., $Q^*f = f \Rightarrow f = \text{const}$. One can define a notion of pointwise equivalence between Markov systems analogous to the one of dynamical systems. This equivalence relation will preserve ergodicity, mixing, and the property of going to equilibrium. For definitions and results concerning measure-preserving Markov processes see Ref. 11, Chap. XIII.

We note that no nontrivial reversible dynamical system goes to equilibrium and that a reversible Markov system (Ω, α, μ, Q), i.e., such that there exists a transition probability $R: \Omega \times \alpha \to [0, 1]$ satisfying $RQ = QR = 1$ on $\mathscr{M}_+ = \{f \in \mathscr{M} : f \geqslant 0\}$, is necessarily induced by a reversible dynamical system. One also has that a unitary operator $V: L^2(\mu) \to L^2(\mu)$ such that $V1 = 1$, $V \geqslant 0$, is induced by a μ-preserving transformation T: $Vf = f \circ T$ (see Ref. 10).

2.2. Deformations and Intrinsically Random Dynamical Systems

I: We shall say that the Markov system $(\Omega', \alpha', \mu', Q')$ is a $(\Lambda, \mathscr{D})$-deformation of the system (Ω, α, μ, Q) if there exists a bounded linear operator $\Lambda: L^2(\mu) \to L^2(\mu')$, and a dense linear subspace $\mathscr{D} \subset L^2(\mu)$ such that

I_1: $\Lambda Q^* = Q'^*\Lambda$, $\qquad$ I_2: Λ injective in $\mathscr{D}$

I_3: $\Lambda 1 = 1$, $\quad \Lambda^* 1 = 1$; $\qquad$ I_4: $\Lambda \geqslant 0$

I_5: $\mathscr{D}' = \Lambda(\mathscr{D})$ dense in $L^2(\mu')$

These properties imply Λ^* injective, positive, $\operatorname{Ran}\Lambda^*$ dense, $\|\Lambda\| = \|\Lambda^*\| = 1$, $Q\Lambda^* = \Lambda^* Q'$. Since the probability spaces are Lebesgue and Λ^*

satisfies the hypothesis of Lemma 1, we can associate to this operator a transition probability Q_{Λ^*} from (Ω, α) to (Ω', α') and one has the conjugation relation $QQ_{\Lambda^*}f = Q_{\Lambda^*}Q'f$ μ-a.e. $f \in \mathscr{M}'_+$. We can find a set of full measure Ω_1 such that $Q_{\Lambda^*}Q' = QQ_{\Lambda^*}$ as equality of transition probabilities from $(\Omega_1, \alpha|_{\Omega_1})$ to (Ω', α'). One has $Q_{\Lambda^*}(Q\tau) = Q'(Q_{\Lambda^*}\tau)$, $\tau \in \rho(\Omega, \alpha)$, and also $Q_{\Lambda^*}(Q^n\tau) = Q'^n(Q_{\Lambda^*}\tau)$, $n \geqslant 0$. One can easily show that if $(\Omega', \alpha', \mu', Q')$ is a $(\Lambda, \mathscr{D})$-deformation of (Ω, α, μ, Q), then Q'^* ergodic implies Q^* ergodic and Q'^* is mixing iff Q^* is mixing. We restrict ourselves now to the case of deformations in the same probability space.

Definition 2. The Markov system $(\Omega, \alpha, \mu, Q')$ is a $\mathscr{D}$-deformation of (Ω, α, μ, Q) if $\mathscr{D}$ is a dense linear subspace in $L^2(\mu)$ and if there exists a bounded linear operator $\Lambda : L^2(\mu) \to L^2(\mu)$ satisfying I_1 to I_5.

Definition 3. We shall say that a dynamical system (Ω, α, μ, T) is intrinsically random if it admits a $\mathscr{D}$-deformation (Ω, α, μ, Q) going to equilibrium.[4]

One can easily show that a reversible dynamical system which is intrinsically random is mixing. If we restrict ourselves to $\mathscr{D}' = L^2(\mu')$ in I_5 then Λ would be a bijection with Λ^{-1} bounded and only the trivial dynamical systems (α trivial) would be intriniscally random.

Theorem 1. A K-system contains a $\mathscr{D}$-deformation going to equilibrium. Consequently it is intrinsically random.

This theorem is proved in Refs. 1 and 2 for Bernoulli shifts and in Ref. 4 for K-systems. We recall here the main steps of the construction of the $\mathscr{D}$-deformation.

We use the notations of Section 2.1. Let (Ω, α, μ, T) be a K-system, $Uf = f \circ T$ and α_n the sequence of σ-algebras in formula (2.2). We construct the operators (E_n in the projector on H_n)

$$\Lambda = \lambda_n E_n + P_0, \qquad W = \left(\sum_{n \in Z} \nu_n E_n + P_0 \right) U \tag{2.7}$$

In (2.7) the sequence $(\lambda_n)_{n \in Z}$ is strictly decreasing in $(0, 1)$, $\lim_{n \to -\infty} \lambda_n = \lambda_{-\infty} = 1$, $\lim_{n \to \infty} \lambda_n = \lambda_\infty = 0$, and such that the sequence $(\nu_n)_{n \in Z}$, $\nu_n = \lambda_n^{-1} \lambda_{n+1}$, is also strictly decreasing. The sequence $\lambda_n = (1 + a^n)^{-1}$, $a > 1$, satisfies all these requirements and $\nu_\infty = \lim_{n \to \infty} \nu_n = a^{-1} < 1$. One easily checks that W satisfies all the properties for V in Lemma 1 and consequently W induces a Markov system $(\Omega, \alpha, \mu, Q_w)$. Taking $\mathscr{D} = \{f\colon \sum_{n \in I} f_n + b,\ f_n \in H_n,\ b \in \mathbb{R},\ I \text{ finite}\}$ as the dense linear subspace in $L^2(\mu)$, one can verify that Λ satisfies all the properties in Definition 2, in particular the conjugation relation I_1 which here reads

$\Lambda U^* = W^*\Lambda$ and consequently the Markov system $(\Omega, \alpha, \mu, Q_W)$ with transition probability $Q_W(\omega, A) = W\xi_A(\omega)$ is a $\mathscr{D}$-deformation of the K-system (Ω, α, μ, T). We note that $U^*\mathscr{D} = U\mathscr{D} = \mathscr{D}$ and that $\Lambda : \mathscr{D} \to \mathscr{D}$ is a bijection.

One has for t integer, $t \geqslant 0$, that

$$W^{*t} = U^{*t}\left(\sum_{n \in Z} v_n^{(t)} E_n + P_0\right), \qquad v_n^{(t)} = \lambda_n^{-1}\lambda_{n+t}$$

and using the property of strict decreasing of (λ_n) one proves that $\| W^{*t} f_0 - P_0 f_0 \| \downarrow_{t \to \infty} 0$, for any density $f_0 \in L^2(\mu) - K_0$, i.e., the Markov system $(\Omega, \alpha, \mu, Q_W)$ converges to equilibrium. Since the inverse dynamical system $(\Omega, \alpha, \mu, T^{-1})$ of a K-system is also a K-system we can also construct in an analogous way an irreversible Markov process which is a $\mathscr{D}$-deformation of T^{-1} and which goes to equilibrium [using the partition α_0' wich satisfies (2.1) for T^{-1}].

3. TRANSITION PROBABILITIES OF $\mathscr{D}$-DEFORMATIONS OF K-SHIFTS

Let (Ω, d) be a compact metric space and $T: \Omega \to \Omega$ a homeomorphism. The stable (unstable) manifolds are the sets

$$\Omega_T^{\mathrm{st}}(\omega) = \{\omega' \in \Omega : d(T^n\omega', T^n\omega) \xrightarrow[n \to \infty]{} 0\} \tag{3.1}$$

$$\Omega_T^{\mathrm{un}}(\omega) = \{\omega' \in \Omega : d(T^{-n}\omega', T^{-n}\omega) \xrightarrow[n \to \infty]{} 0\} \tag{3.2}$$

One has $\Omega_{T^{-1}}^{\overset{\mathrm{st}}{\mathrm{un}}}(\omega) = \Omega_T^{\overset{\mathrm{un}}{\mathrm{st}}}(\omega)$. In the case of the shift $\sigma : X \to X$ these manifolds are Borel sets in X:

$$X_\sigma^{\mathrm{st}}(x) = \{y \in X : y_i = x_i, \forall i \geqslant k(y) \text{ for some } k(y)\}, \qquad x \in X \quad (3.1')$$

$$X_\sigma^{\mathrm{un}}(x) = \{y \in X : y_i = x_i, \forall i \leqslant k(y) \text{ for some } k(y)\}, \qquad x \in X \quad (3.2')$$

We consider now K-shifts. According to Theorem 1 we can construct a $\mathscr{D}$-deformation going to equilibrium with transition probability Q_w induced by W given by (2.7). We prove the following:

Theorem 2. Let $(X, \mathscr{B}, \mu, \sigma)$ be a K-shift. Then it admits a $\mathscr{D}$-deformation $(X, \mathscr{B}, \mu, Q_w)$ succh that (a) $Q_w(x, X_\sigma^{\mathrm{st}}(\sigma x)) = 1$, $x \in X$; (b) Q_w has a point mass if $v_\infty > 0$. For Bernoulli shifts it is concentrated on the transformed point by the shift: $Q_w(x, \{\sigma x\}) = v_\infty$.

Proof. From (2.7) and using $E_n = R_n - R_{n-1}$ one has

$$W = \left(\sum \bar{\nu}_n R_n + \nu_\infty R_\infty\right) U, \bar{\nu}_n = \nu_n - \nu_{n+1}$$

$R_\infty = E^{\mathscr{B}_\infty}$ is the identity operator. The operators $W_n \equiv R_{n-1} U$ satisfy all the properties of V in Lemma 1 and consequently induce Markov systems $(\Omega, \mathscr{B}, \mu, Q_{W_n})$, $n \in Z$, with transition probabilities $Q_n = Q_{W_n}$ given by $Q_n(x, A) = W_n \xi_A(x)$. One has $U\xi_{B(k,m)(y)} = \xi_{B(k+1,m+1)(\sigma^{-1}y)}$, and using for R_n the version (2.4) one obtains for the infinite cylinder $B(k, \infty)(y)$ the result $Q_{n+1}(x, B(k, \infty)(y)) = \xi_{B(k+1,\infty)(\sigma^{-1}y)}(x)$, $n \geqslant -(k+1)$. Then for $y = \sigma x$ one has $Q_{n+1}(x, B(-(n+1), \infty)(\sigma x)) = 1$, $n \in Z$. Putting $Q_\infty(x, A) = U\xi_A(x)$, $A \in \mathscr{B}$, one obtains the transition probability $Q_\infty(x, \{\sigma x\}) = 1$. From the previous expression of W we conclude that $Q_w(x, A) = W\xi_A(x)$ can be written as

$$Q_W(x, \cdot) = \sum \bar{\nu}_n Q_{n+1}(x, \ ) + \nu_\infty Q_\infty(x, \ ) \tag{3.3}$$

One has $X_\sigma^{\text{st}}(\sigma x) = \bigcup_{n \in Z} B(n, \infty)(\sigma x)$, and consequently $Q_W(x, X_\sigma^{\text{st}}(\sigma x)) = 1$, which is part (a) of the theorem. On the other hand we can compute directly $Q_W(x, B(k, m)(\sigma x)) = W\xi_{B(k,m)(\sigma x)}(x)$ using (2.4) to (2.4″). One obtains

$$\begin{aligned} Q_W(x, B(k, m)(\sigma x)) = &\sum_{n < -(m+1)} \bar{\nu}_n R_n \xi_{B(k+1,m+1)(x)}(x) \\ &+ \sum_{n=-(m+1)}^{n=-(k+2)} \bar{\nu}_n R_n \xi_{B(k+1,-(n+1))(x)}(x) + \nu_{-(k+1)} \end{aligned} \tag{3.4}$$

When $m \to \infty$ one shows that the first sum goes to zero and

$$\begin{aligned} Q_W(x, B(k, \infty)(\sigma x)) = \nu_{-(k+1)} + &\sum_{n=1}^{\infty} \bar{\nu}_{-(k+n+1)} R_{-(k+n+1)} \\ &\times \xi_{B(k+1,k+n)(x)}(x) \end{aligned} \tag{3.5}$$

If $k \to -\infty$ the sum in (3.5) is non-negative and since $B(-\infty, \infty)(\sigma x) = \{\sigma x\}$ we obtain $Q_W(x, \{\sigma x\}) \geqslant \nu_\infty$. In an analogous way it is easy to see that σx is the only point with ν_∞ for Bernoulli shifts.

In the case of Bernoulli shifts we can give an explicit form for a natural version of the transition probability which is obtained using formulas (2.4)–(2.4‴) for R_n. One obtains

$$\begin{aligned} W\xi_{B(k,m)(y)} = &\sum_{n=-(m+1)}^{-(k+2)} \bar{\nu}_n \xi_{B(-n,m+1)(\sigma^{-1}y)} \mu_\pi(B(k, -(n+2))(y)) \\ &+ \nu_{-(k+1)} \xi_{B(k+1,m+1)(\sigma^{-1}y)} + (1 - \nu_{-(m+1)}) \mu_\pi(B(k, m)(y)) \end{aligned} \tag{3.6}$$

and consequently [compare with (3.4)]

$$Q_W(x, B(k, m)(\sigma x)) = \sum_{n=-(m+1)}^{-(k+2)} \bar{\nu}_n \mu_\pi(B(k, -(n+2))(\sigma x)) + \nu_{-(k+1)} + (1 - \nu_{-(m+1)}) \mu_\pi(B(k, m)(\sigma x)) \quad (3.7)$$

For a shift we can define the fibers $\hat{X}_k^{\text{st}}(y) = \{z : z_j = y_j,\ k+1 \leqslant j,\ z_k \neq y_k\}$ and the stable manifold of y is the disjoint union $X_\sigma^{\text{st}}(y) = [\bigcup_{k \in Z} X_k^{\text{st}}(y)] \cup \{y\}$. We consider the sets $B_i^k(k-r, m)(y) = \{z : z_{k-r} = i_r, ..., z_k = i_0 \neq y_k,\ z_j = y_j,\ k+1 \leqslant j \leqslant m\}$ which are cylinders $B(k-r, m)$ $(i_r, ..., i_0, y_{k+1}, ..., y_m)$. Using (3.6) one obtains when $m \to \infty$ for Bernoulli shifts

$$Q_W(x, B_i^k(k-r, \infty)(\sigma x)) = \pi_{i_0} \cdots \pi_{i_r} \pi_{k+1}(x)^{-1} \sum_{n=k+2}^{\infty} \bar{\nu}_{-n} \pi_{k+1}(x) \cdots \pi_{n-1}(x) \quad (3.8)$$

with the notation $\pi_l(x) \equiv \pi_{x_l}$. We have

$$\hat{X}_k^{\text{st}}(y) = \bigcup_{\substack{i_1, ..., i_n \\ i_0 \neq y_k}} B_i^k(k-r, \infty)(y)$$

a disjoint union; then

$$Q_W(x, \hat{X}_k^{\text{st}}(\sigma x)) = (\pi_{k+1}(x)^{-1} - 1) \sum_{n=k+2}^{\infty} \bar{\nu}_{-n} \pi_{k+1}(x) \cdots \pi_{n-1}(x)$$

and a simple calculation shows $Q_W(x, \hat{X}_k^{\text{st}}(\sigma x)) = 1 - \nu_\infty$. This provides an alternative proof of Theorem 2 since from (3.5) we see (taking the limit $k \to -\infty$, $m \to \infty$) that $Q_W(x, \{\sigma x\}) = \nu_\infty$. This proof also applies for K-shifts and was given by us in Ref. 7 for the Baker system.

Let us consider now in $\hat{X}_k^{\text{st}}(y)$ the σ-algebra $\bar{\mathscr{B}}_k(y)$ generated by the sets $B_i^k(k-r, \infty)(y)$. In the space $(\hat{X}_k^{\text{st}}(y), \bar{\mathscr{B}}_k(y))$ the measure μ_π induces a measure μ_k defined by $\mu_k(B_i^k(k-r, \infty)(y)) = \pi_{i_0} \cdots \pi_{i_r}$.

On the other hand the measure $Q_W(x, \cdot)$ induces in $(\hat{X}_k^{\text{st}}(\sigma x), \bar{\mathscr{B}}_k(\sigma x))$ a measure $Q_W^k(x, \cdot) = Q_W(x, \cdot \cap \hat{X}_k^{\text{st}}(\sigma x))$. Defining [see (3.8)]

$$h(y, k) = \pi_k(y)^{-1} \sum_{n=k+2}^{\infty} \bar{\nu}_{-n} \pi_k(y) \cdots \pi_{n-2}(y)$$

we have the following proposition.

Proposition 1. For Bernoulli shifts the part without point mass of the transition probability Q_W is decomposed along the fibers $\{X_k^{\text{st}}(\sigma x)\}_{k \in Z}$. If

Q_W is the measure induced in the kth fiber, it is absolutely continuous with respect to the measure μ_k in $(\hat{X}_k^{st}(x), \bar{\mathscr{B}}_k(\sigma x))$, and one has

$$\frac{dQ_W^k(x, \cdot)}{d\mu_k}(z) = h(\sigma x, k), \qquad z \in \hat{X}_k^{st}(\sigma x)$$

4. MEASURES CONVERGING WEAKLY TO EQUILIBRIUM IN BERNOULLI SHIFTS

Let (Ω, α, μ) be a probability space with Ω a compact metric space and α the Borel σ-algebra. We denote by ∂A the boundary of the set $A \subset \Omega$. A sequence of probability measures $(\nu_n)_{n \geqslant 1}$ on (Ω, α) converges weakly to μ; we write $\nu_n \xrightarrow{w}_{n\to\infty} \mu$, if $\nu_n(A) \to \mu(A)$, $\forall A \in \alpha$ such that $\mu(\partial A) = 0$. The set of measures which converge weakly to μ is convex. A sufficient condition for $\nu_n \xrightarrow{w}_{n\to\infty} \mu$ is $\nu_n(A) \to \mu(A)$ for any set A in a family $\alpha_0 \subset \alpha$ verifying the following condition (E): it is closed by intersection, generates α, and for any ball $B(\omega, \varepsilon) \subset \Omega$ there exists $A \in \alpha_0$, $A \subset B(\omega, \varepsilon)$ with ω interior point of A (see Corollary 1 of Theorem 2.2 in Ref. 9). We remark that in the case $(X, \mathscr{B})$ the set of cylinders ζ satisfies (E) and consequently in that space $\nu_n \xrightarrow{w}_{n\to\infty} \mu$ if $\nu_n(B) \to_{n\to\infty} \mu(B)$, $\forall B \in \zeta$.

Definition 4. Let (Ω, α, μ, Q) be a Markov system. We shall say that the probability measure τ on (Ω, α) converges to the equilibrium μ by the action of Q; we write $\tau \in \mathrm{Eq}(\mu, Q)$, if $Q^n \tau \xrightarrow{w}_{n\to\infty} \mu$.

Then $\mathrm{Eq}(\mu, Q)$ is convex. We remark that $(Q^n\tau)(A) = \int \tau(d\omega)\, Q^{(n)}(\omega, A) = \int \tau(d\omega)\, Q^n \xi_A(\omega)$, where $Q^{(n)}(\omega, A)$ is the transition probability from ω to A in n steps. If τ is absolutely continuous with respect to μ, $\tau \ll \mu$, and if $d\tau/d\mu \in L^2(\mu)$ one has $Q^n\tau(A) = \langle d\tau/d\mu, Q^n \xi_A \rangle = \langle Q^{*n}\, d\tau/d\mu, \xi_A \rangle$, which goes to equilibrium $\mu(A)$ if Q is mixing and then $\tau \in \mathrm{Eq}(\mu, Q)$.

Let $\mathscr{D}(\alpha_0) = \{\sum_{n \in I} a_n \xi_{A_n} + b:\ a_n \in \mathbb{R},\ b \in \mathbb{R},\ A_n \in \alpha_0, I \text{ finite}\}$ where α_0 verify condition (E); then $\mathscr{D}(\alpha_0)$ is dense in $L^2(\mu)$. We suppose that $\mathscr{D} \supset \mathscr{D}(\alpha_0)$ and $Q_{\Lambda^*} \mathscr{D}(\alpha_0) \subset \mathscr{D}(\alpha_0)$ (which is the case for Bernoulli shifts); then we have the following proposition.

Proposition 2. If Q' is a $\mathscr{D}$-deformation of Q the Markov process Q_{Λ^*} satisfies $Q_{\Lambda^*}(\mathrm{Eq}(\mu, Q)) \subset \mathrm{Eq}(\mu, Q')$.

Proof. It is enough to prove that if $\tau \in \mathrm{Eq}(\mu, Q)$ then $(Q'^n(Q_{\Lambda^*}\tau))(\xi_A) \to_{n\to\infty} \mu(A)$ for $A \in \alpha_0$. From (2.8) we have $Q'^n(Q_{\Lambda^*}\tau) = Q_{\Lambda^*}(Q^n\tau)$ and from (2.6) $Q_{\Lambda^*}(Q^n\tau)(\xi_A) = (Q^n\tau)(Q_{\Lambda^*}\xi_A)$ which converges to $\mu(Q_{\Lambda^*}\xi_A)$ since $Q_{\Lambda^*}\xi_A \in \mathscr{D}(\alpha_0)$. Moreover $\mu(Q_{\Lambda^*}\xi_A) = \langle \Lambda^*\xi_A, 1 \rangle = \mu(A)$.

If (Ω, α, μ, T) is a dynamical system a probability measure τ converges to the equilibrium μ by Q_T iff

$$\tau(T^n A) \xrightarrow[n\to\infty]{} \mu(A), \qquad \forall A \in \alpha, \qquad \mu(\partial A) = 0 \tag{4.1}$$

or on the sets $A \in \alpha_0$, with α_0 verifying (E). We treat now the case of the Bernoulli shift $(X, \mathscr{B}, \mu, \sigma)$. We recall that

$$W^t = \left(\sum_{n \in Z} v_n^{(t)} E_n + P_0 \right) U^t, \qquad t \in N, \qquad v_n^{(t)} = \lambda_n^{-1} \lambda_{n+t}$$

Theorem 3. $\mathrm{Eq}(Q_0, \mu_\pi) = \mathrm{Eq}(Q_W, \mu_\pi)$, i.e., a probability measure τ on $(X, \mathscr{B})$ converges to the equilibrium μ_π by the Markov system Q_W iff it converges to the equilibrium μ_π by the dynamical system Q, which means that

$$\tau(\sigma^{-n} B) \xrightarrow[n\to\infty]{} \mu_\pi(B), \qquad \forall B \in \zeta \tag{4.1$'$}$$

Proof. Let $B = B(k, m)$ be a cylinder; then

$$Q_W^{(t)}(x, B) = W^t \xi_B(x) = \sum_{r=1}^{l} \bar{v}^{(t)}_{-(k+r+t)} \mu_\pi(B(k, k+r-1))\, \xi_{\sigma^{-t}B(k+r,\, m)}(x)$$
$$+ v^{(t)}_{-(k+t)} \xi_{\sigma^{-t}B} + (1 - v^{(t)}_{-(m+t)})\, \mu_\pi(B)$$

and

$$(Q_W^t \tau)(B) = \sum_{r=1}^{l} \bar{v}^{(t)}_{-(k+r+t)} \mu_\pi(B(k, k+r-1))\, \tau(\sigma^{-t} B(k+r, m))$$
$$+ v^{(t)}_{-(k+t)} \tau(\sigma^{-t} B) + (1 - v^{(t)}_{-(m+t)})\, \mu_\pi(B) \tag{4.2}$$

One has $v^{(t)}_{n-t} \to_{t\to\infty} \lambda_n$, $\bar{v}^{(t)}_{n-t} \to_{t\to\infty} \bar{\lambda}_n$. We shall prove the result by recursion on $l = m - k \geqslant 0$. If $l = 0$, $B = B(k, k)$, and

$$(Q_W^t \tau)(B) = v^{(t)}_{-(k+t)} \tau(\sigma^{-t} B(k, k))\, \mu_\pi(B(k, k)) + (1 - v^{(t)}_{-(k+t)})\, \mu_\pi((k, k))$$

which converges to $\mu_\pi(B)$ iff $(Q_\sigma^t \tau)(B) = \tau(\sigma^{-t} B) \to_{n\to\infty} \mu_\pi(B)$. We suppose now that for all cylinders such that $l' \leqslant l - 1$ the result is verified, and we shall prove it for $l' = l$. Since $r > 1$ by the hypothesis of recursion we have

$$\mu_\pi(B(k, k+r-1))\, \tau(\sigma^{-t} B(k+r, m)) \xrightarrow[r\to\infty]{} \mu_\pi(B)$$

Using

$$\sum_{r=1}^{l} \lambda_{-(k+r)} + (1 - \lambda_{-m}) = 1 - \lambda_{-k} \quad \text{and} \quad v^{(t)}_{n-t} \xrightarrow[t\to\infty]{} \lambda_n$$

we conclude

$$Q_W^t \tau(B) \xrightarrow[t \to \infty]{} \mu_\pi(B) \qquad \text{iff} \quad \tau(\sigma^{-t} B(k, m)) \xrightarrow[n \to \infty]{} \mu_\pi(B(k, m))$$

Then

$$\tau \in \mathrm{Eq}(Q_W, \mu_\pi) \qquad \text{iff} \quad \tau \in \mathrm{Eq}(Q_\sigma, \mu_\pi)$$

Defining uniform probability measures with respect to μ_π on the unstable fibers $\hat{X}^{\mathrm{un}}{}_k^n(x)$ one obtains a sequence of measures converging to a Dirac measure concentrated in x when $k \to \infty$. On the other hand, each measure of the sequence for finite k converges to the equilibrium μ_π by the temporal evolution of the system; consequently $\mathrm{Eq}(\mu_\pi, Q_\sigma) = \mathrm{Eq}(\mu_\pi, Q_W)$ is a convex set which is not closed by weak convergence. By direct calculation one proves that the operator Q_Λ acting on the set of probability measures $\rho(X, B)$ is injective (one shows that $Q_\Lambda \tau = Q_\Lambda \tau'$, $\tau(B) = \tau'(B)$, $\forall B \in \zeta$, by recursion on the length of B). But it is not surjective. Let $\tau \in \rho(X, B)$ and put $h(\tau, B) = \tau(B) - (1 - \lambda_{-m}) \mu_\pi(B)$ for any cylinder $B = B(k, m)$. We define $\tau'(B(k, k)) = \lambda_{-k}^{-1} h(\tau, B(k, k))$ and by recursion on the length of B:

$$\tau'(B(k, m)) = \lambda_{-k}^{-1} \left(h(\tau, B(k, m)) - \sum_{r=1}^{l} \tau'(B(k + r, m)) \mu_m(B(k, k + r - 1)) \lambda_{-(k+r)} \right)$$

One can now prove that $\tau \in Q_\Lambda(\rho(X, B))$ iff $\tau'(B(k, m)) \geqslant 0$, $\forall k \leqslant m$. It is easy to prove that τ' defines a probability measure on $(X, \mathscr{B})$ and by construction $Q_\Lambda \tau' = \tau$. Then $Q_\Lambda(\rho(X, \mathscr{B}))$ is strictly contained in the set of measures with support in X and Q_Λ is not surjective. In fact it is enough to take for τ a probability measure induced by a density of $L^2(\mu_\pi)$ which vanishes in a fixed cylinder. This example also shows that $Q_\Lambda(\mathrm{Eq}(Q_\sigma, \mu_\pi)) \nsubseteq \mathrm{Eq}(Q_W, \mu_\pi)$. We see then that although $\mathrm{Eq}(Q_\sigma, \mu_\pi) = \mathrm{Eq}(Q_W, \mu_\pi)$ one cannot pass from one set to the other by the canonical operator induced by Λ. A study of the convergence of measures in relation to the definition of entropy functionals is given in Ref. 12.

ACKNOWLEDGEMENTS

The authors are especially indebted to Professor S. Goldstein for his valuable suggestions, which have been most important in the final version of this work. They are also grateful to their colleagues of the Faculté des Sciences of the University of Brussels and of the Laboratoire de Probabilités of the University of Paris VI for many interesting and stimulating discussions. One of us (S.M.) also thanks the Universities of Brussels and Paris VI for their hospitality.

REFERENCES

1. B. Misra, I. Prigogine, and M. Courbage, *Physica* **98A**:1 (1979).
2. M. Courbage and B. Misra, *Physica* **104A**:359 (1980).
3. B. Misra, *Proc. Natl. Acad. Sci. U.S.A.* **75**:1627 (1978).
4. S. Goldstein, B. Misra, and M. Courbage, *J. Stat. Phys.* **25**:111 (1981).
5. J. Neveu, *Bases Mathémtiques du Calcul des Probabilités* (Masson, Paris, 1970).
6. V. I. Arnold and A. Avez, *Problèmes ergodiques de la Mécanique Classique* (Gauthier-Villers, Paris, 1967).
7. S. Martinez and E. Tirapegui, Irreversible Evolution of Dynamical Systems, in *Lecture Notes in Physics* 179, Proceedings Sitges 1982, ed. L. Garrido (Springer, Berlin, 1982).
8. S. Martinez and E. Tirapegui, *Phys. Lett.* **95A**:143 (1983).
9. P. Billingsley, *Convergence of Probability Measures* (Wiley, New York, 1968).
10. K. Goodrich, K. Gustafson, and B. Misra, *Physica* **A108**:379 (1980).
11. K. Yosida, *Functional Analysis* (Springer-Verlag, Berlin, 1968).
12. S. Martinez and E. Tirapegui, Entropy functionals associated to shift dynamical systems, *Physica* **122A**:593–601 (1983).